On relevance of modified gravities

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Abstract

It is shown that modified gravity theories with Lagrangian composed of the three quadratic invariants of Riemannian curvature are not appropriate. The field equations are either incompatible and/or irregular (like $f(R)-$gravities), or, if compatible, leads to linear instability of polarizations related to the Weyl tensor.

More relevant modification is the frame field theory, the best and unique variant of Absolute Parallelism; it has no free parameters ($D=5$ is a must) and no singularities arising in solutions. I sketch few features of this theory.

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1 Riemann-squared modified gravities

Let me not to list all papers which mention modified gravity theories with Lagrangian composed of the three invariants quadratic in the Riemannian curvature. To my thinking, all Riemann-squared-gravities (as well as $f(R)-$gravities, $f(R) \neq R$) are not appropriate.

So, $(a + bR + R^2/2)-$gravity (R is the Ricci scalar) leads to incompatible PDE system (the trace part, $\mathbf{E}_\mu{}^\mu = 0$, can be added or subtracted freely):

$$\mathbf{E}_{\mu\nu} = R_{;\mu;\nu} - R_{\mu\nu}(b + R) + g_{\mu\nu}(...R..R^2) = 0;$$

some people prefer to move the principal derivatives (4-th order) to RHS, perhaps trying to hide them in the energy-momentum tensor; however it is not good way to deal with PDEs. The next combination of prolonged equations,

$$\mathbf{E}_{\mu\nu;\lambda} - \mathbf{E}_{\mu\lambda;\nu} = 0,$$

after cancellation of principal derivatives (5-th order), gives new 3-d order equations which are irregular in the second jets: the term

$$R_{;\varepsilon}R^{\varepsilon}{}_{\mu\nu\lambda}$$

can not be cancelled by other terms which contain only Ricci tensor and scalar. The rank of the new subsystem depends on the second derivatives, $g_{\mu\nu,\lambda\rho}$ (for the definition of PDE regularity see [3]).

As a rule, researchers of modified gravities concentrate their efforts on most symmetrical problems including cosmological solutions with the spherical symmetry. In this case the irregularity of the above system is safely masked: the new subsystem becomes just identity due to skew-symmetry of its two indices.

Also irregular in second jets are equations of Gauss-Bonnet (or Lovelock) gravity with extra dimension(s).

The most interesting case, $R_{\mu\nu}G^{\mu\nu}-$gravity (the Ricci tensor is contracted with the Einstein tensor), gives the following compatible system:

$$-\mathbf{D}_{\mu\nu} = G_{\mu\nu;\lambda}{}^{;\lambda} + G^{\epsilon\tau}(2R_{\epsilon\mu\tau\nu} - \tfrac{1}{2}g_{\mu\nu}R_{\epsilon\tau}) = 0; \;\; \mathbf{D}_{\mu\nu;\lambda}g^{\nu\lambda} \equiv 0\,. \qquad (1)$$

In linear approximation, there are simple evolution equations for Ricci tensor and scalar (total $D(D-1)/2$ polarizations—including one scalar polarization):

$$\Box R = 0, \ \Box R_{\mu\nu} = 0.$$

Using the Bianchi identity, $R_{\mu\nu[\lambda\epsilon;\tau]} \equiv 0$, its prolongation and contractions,

$$R_{\mu\nu[\lambda\epsilon;\tau];\rho} g^{\tau\rho} \equiv 0, \ R_{\mu\nu[\lambda\epsilon;\tau]} g^{\mu\tau} \equiv 0,$$

one can write the evolution equation for the Riemann (or Weyl) tensor (in linear approximation again):

$$\Box R_{\mu\epsilon\nu\tau} = R_{\mu\nu,\epsilon\tau} - R_{\mu\tau,\nu\epsilon} + R_{\epsilon\tau,\mu\nu} - R_{\epsilon\nu,\mu\tau}. \tag{2}$$

This equation is more complex: it has the source term (in its RHS) composed from the Ricci tensor. As a result in general case, when the Ricci-polarizations do not vanish, the polarizations related to the Weyl tensor [and responsible for gravity, tidal forces; their number is $D(D-3)/2$] should grow linearly with time,

$$a(t) = (c_0 + c_1 t) \exp(-i\omega t),$$

while linear approximation is valid.

This means that the regime of weak gravity is linearly unstable, as well as the trivial solution itself (i.e., *nothing is unreal* in this theory). Hence the theory is physically irrelevant—we still live in very weak gravity. (Note that in General Relativity, when the Ricci tensor is expressed through the energy-momentum tensor [which does not expand into plane waves—with dispersion of light in vacuum], the equation (2) defines radiation of gravitation waves.)

The linear instability here does not contradict the correctness of Cauchy problem; the modern compatibility theory (e.g., the Pommaret's book [3]) gives easy answers about the Cauchy problem, number of polarizations, and so on (especially easy for analytical PDE systems).

And the last remark. Let $\mathcal{L} = \sqrt{-g} L$ is a homogeneous Lagrangian density of order p in metrics, that is

$$\mathcal{L}(\kappa g_{\mu\nu}) = \kappa^p \mathcal{L}(g_{\mu\nu}) \, .$$

The result of variation, the symmetric tensor

$$\mathbf{D}^{\mu\nu} = \frac{1}{\sqrt{-g}} \frac{\delta \mathcal{L}}{\delta g_{\mu\nu}}, \ \ \mathbf{D}^{\mu\nu}{}_{;\nu} \equiv 0,$$

has the next relation to its Lagrangian (the trace is proportional to the Lagrangian scalar up to a covariant divergence):

$$\mathbf{D}^{\mu\nu} g_{\mu\nu} = p\, L + A^{\mu}{}_{;\mu}.$$

If $\mathbf{D}_{\mu\nu}$ (with $\mathbf{D}^{\mu\nu}{}_{;\nu} \equiv 0$) is found just by using the Bianchi identity, this relation reveals the corresponding Lagrangian, see eg. equation (1).

2 Frame field theory

The theory of frame field, $h^a{}_\mu$, also known as Absolute Parallelism (AP), has large symmetry group which includes both global symmetries of Special Relativity (this defines the signature) and local symmetries, the pseudogroup *Diff*(D), of General Relativity.

AP is more appropriate as a modified gravity, or just a good theory with topological charges and quasi-charges (their phenomenology, at some conditions and to a certain extent, can look like a quantum field theory) [2]. In this case, the Ricci tensor has very specific form (due to field equations; linear approximation):

$$R_{\mu\nu} \propto \Phi_{(\mu,\nu)}, \quad \Phi_\mu = h_a{}^\nu (h^a{}_{\nu,\mu} - h^a{}_{\mu,\nu}) \text{ – trace of torsion;}$$

this form does not cause the Weyl polarizations growth, see (1); so the weak gravity regime (but not the trivial solution!) is stable.

2.1 Co- and contra-singularities and unique equation

There is one unique equation of AP (non-Lagrangian, with the unique D) which solutions are free of arising singularities. The formal integrability test [1] can be extended to the cases of degeneration of either co-frame matrix, $h^a{}_\mu$ (co-singularities), or contra-variant frame (or density of some weight), serving as a local and covariant test for singularities. This test singles

out the next equation (and $D{=}5$ [2]; $\eta_{ab}=\mathrm{diag}(-1,1,\dots,1)$, then $h=\det h^a{}_\mu=\sqrt{-g}$):

$$\mathbf{E}_{a\mu}=L_{a\mu\nu;\nu}-\tfrac{1}{3}(f_{a\mu}+L_{a\mu\nu}\Phi_\nu)=0\,; \tag{3}$$

here $$L_{a\mu\nu}=L_{a[\mu\nu]}=\Lambda_{a\mu\nu}-S_{a\mu\nu}-\tfrac{2}{3}h_{a[\mu}\Phi_{\nu]},$$

$$\Lambda_{a\mu\nu}=2h_{a[\mu,\nu]},\;\; S_{\mu\nu\lambda}=3\Lambda_{[\mu\nu\lambda]},\;\; \Phi_\mu=\Lambda_{aa\mu},\;\; f_{\mu\nu}=2\Phi_{[\mu,\nu]}.$$

Coma "," and semicolon ";" denote partial derivative and covariant differentiation with symmetric Levi-Civita connection, respectively. One should retain the identities:

$$\Lambda_{a[\mu\nu;\lambda]}\equiv 0\,,\quad h_{a\lambda}\Lambda_{abc;\lambda}\equiv f_{cb}\,(=f_{\mu\nu}h^\mu_c h^\nu_b),\;\; f_{[\mu\nu;\lambda]}\equiv 0. \tag{4}$$

Equation $\mathbf{E}_{a\mu;\mu}=0$ gives 'Maxwell-like equation' (I omit η_{ab} and $g^{\mu\nu}=h^\mu_a h^\nu_a$ in contractions):

$$(f_{a\mu}+L_{a\mu\nu}\Phi_\nu)_{;\mu}=0,\;\text{ or }\;\; f_{\mu\nu;\nu}=(S_{\mu\nu\lambda}\Phi_\lambda)_{;\nu}\;\;(=-\tfrac{1}{2}S_{\mu\nu\lambda}f_{\nu\lambda},\text{ see below})\,. \tag{5}$$

Really (5) follows from the symmetric part, because skewsymmetric one gives the identity; the trace part becomes irregular (principal derivatives vanish) if $D=4$ (forbidden D):

$$2\mathbf{E}_{[\nu\mu]}=S_{\mu\nu\lambda;\lambda}=0,\;\; \mathbf{E}_{[\nu\mu];\nu}\equiv 0;\quad \mathbf{E}_{\mu\mu}=\mathbf{E}_{a\mu}h^\mu_b\eta^{ab}=\tfrac{4-D}{3}\Phi_{\mu;\mu}+(\Lambda^2)=0.$$

System (3) remains compatible under adding $f_{\mu\nu}=0$, see (5); this is not the case for other covariants, S,Φ, or Riemannian curvature, which relates to Λ as usually:

$$R_{a\mu\nu\lambda}=2h_{a\mu;[\nu;\lambda]};\;\; h_{a\mu}h_{a\nu;\lambda}=\tfrac{1}{2}S_{\mu\nu\lambda}-\Lambda_{\lambda\mu\nu}.$$

GR is a special case of AP. Using 3-minors (ie., co-rank 3) of co-metric,

$$[\mu\nu,\varepsilon\tau,\alpha\beta]\equiv\partial^3(-g)/(\partial g_{\mu\nu}\partial g_{\varepsilon\tau}\partial g_{\alpha\beta}),$$

and their skew-symmetry, one can write the vacuum GR equation as follows: $2(-g)G^{\mu\nu}=$

$$[\mu\nu,\varepsilon\tau]_{,\varepsilon\tau}+(g'^2)=[\mu\nu,\varepsilon\tau,\alpha\beta](g_{\alpha\beta,\varepsilon\tau}+g^{\rho\phi}\Gamma_{\rho,\varepsilon\tau}\Gamma_{\phi,\alpha\beta})=[\mu\nu,\varepsilon\tau,\alpha\beta]R_{\alpha\beta\varepsilon\tau}=0. \tag{6}$$

Similarly, all AP equations can be rewritten that 2-minors of co-frame,

$$\begin{pmatrix}\mu\;\nu\\ a\;b\end{pmatrix}=\frac{\partial^2 h}{\partial h^a{}_\mu\partial h^b{}_\nu}=2!\,hh^\mu_{[a}h^\nu_{b]}\;\;\text{ (i.e. }\;[\mu_1\nu_1,\dots,\mu_k\nu_k]=\tfrac{1}{k!}\begin{pmatrix}\mu_1\cdots\mu_k\\ a_1\cdots\alpha_k\end{pmatrix}\begin{pmatrix}\nu_1\cdots\nu_k\\ a_1\cdots\alpha_k\end{pmatrix}\text{)},$$

completely define the coefficients at the principal derivatives. The simplest compatible equation (see Einstein–Mayer classification of compatible equations in $4D$ AP [3]),

$$\mathbf{E}^*{}_{a\mu} = \Lambda_{a\mu\nu;\nu} = 0, \ \mathbf{E}^*{}_{a\mu;\mu} \equiv 0 \tag{7}$$

gives

$$h^2 \mathbf{E}^*{}_a{}^\mu = (h_{a\alpha,\beta\nu} - h_{a\beta,\alpha\nu})(-g)g^{\alpha\mu}g^{\beta\nu} + (h'^2) = h_{a\alpha,\beta\nu}[\alpha\mu, \beta\nu] + (h'^2) \ .$$

Like determinant, k-minors ($k \le D$) are multi-linear expressions in elements of $h^a{}_\mu$-matrix, and some minors do not vanish when rank $h^a{}_\mu = D - 1$.

For any AP equation [including Eqs. (6) and (7)], with the *unique exception*, Eq. (3), (where only skew-symmetric part participates in identity and can be written with 2- and 3-minors, while symmetric part needs 1-minors vanishing too rapidly), the regularity of principal terms survives (and symbol G_2 keeps involutive) if rank $g_{\mu\nu} = D - 1$.

This observation is important and relevant to the problem of singularities; it means seemingly that the unique equation (3) does not suffer of nascent co-singularities in solutions of general position.

The other case is contra-singularities [2] relating to degeneration of contra-variant density of some weight:

$$H_a{}^\mu = h^{1/D_*} h_a{}^\mu; H = \det H^a{}_\mu, \ h_a{}^\mu = H^{1/(D-D_*)} H_a{}^\mu \, . \tag{8}$$

Here D_* depends on equation: $D_* = 2$ for GR, $D_* = \infty$ for Eq. (7), and $D_* = 4$ for the unique equation (which can be written 3-linearly in $H_a{}^\mu$ and its derivatives [2]).

If integer, D_* is the forbidden spacetime dimension. The nearest possible $D = 5$ is of special interest: in this case minor $H^{-1}H^a{}_\mu$ simply coincides with $h^a{}_\mu$; that is, contra-singularity simultaneously implies co-singularity (of high co-rank), but that is impossible! The possible interpretation of this observation is: for the unique equation, contra-singularities are impossible if $D = 5$ (perhaps due to some specifics of *Diff*-orbits on $H_a{}^\mu$-space). This leaves no room for changes in the theory (assuming Nature does not like singularities).

2.2 Tensor $T_{\mu\nu}$ and post-Newtonian effects (Pauli's questions to AP)

One might rearrange $\mathbf{E}_{(\mu\nu)} = 0$ picking out (into LHS) the Einstein tensor, $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$, but the rest terms are not proper energy-momentum tensor: they contain linear terms $\Phi_{(\mu;\nu)}$ (no positive energy (!); instead one more presentation of 'Maxwell equation' (5) is possible—as divergence of symmetrical tensor).

However, the prolonged equation $\mathbf{E}_{(\mu\nu);\lambda;\lambda} = 0$ can be written as RG-gravity:

$$G_{\mu\nu;\lambda;\lambda} + G_{\epsilon\tau}(2R_{\epsilon\mu\tau\nu} - \tfrac{1}{2} g_{\mu\nu} R_{\epsilon\tau}) = T_{\mu\nu}(\Lambda'^2, \cdots),\ T_{\mu\nu;\nu} = 0; \tag{9}$$

up to quadratic terms, $\quad T_{\mu\nu} = \frac{2}{9}(\frac{1}{4} g_{\mu\nu} f^2 - f_{\mu\lambda} f_{\nu\lambda}) + A_{\mu\epsilon\nu\tau}(\Lambda^2)_{,\epsilon\tau}$; tensor A has symmetries of Riemann tensor, so the term A'' adds nothing to momentum and angular momentum.

It is worth noting that:

(a) the theory does not match GR, but reveals RG-gravity (sure, (9) does not contain all);

(b) only f-component (three transverse polarizations in $D = 5$) carries D-momentum and angular momentum ('powerful' waves); other 12 polarizations are 'powerless', or 'weightless'. This is a very unusual feature—impossible in Lagrangian tradition; how to quantize ?

(c) f-component feels only metric and S-field, see (5), but S has effect only on polarization of f: $S_{[\mu\nu\lambda]}$ does not enter eikonal equation, and f moves along usual Riemannian geodesic; trace $T_{\mu\mu} = \frac{1}{18} f_{\mu\nu} f_{\mu\nu}$ can be non-zero if $f^2 \neq 0$;

(e) it should be stressed that f-component is not usual (quantum) EM-field—just important covariant responsible for energy-momentum (there is no gradient invariance for f; phenomenological quantum fields should account somehow for topological (quasi)charges [2]).

Another strange feature is the instability of trivial solution: some 'powerless' polarizations grow linearly with time in presence of 'powerful' f-polarizations. Really, the linearized Eq. (3) and identity (4) give (following equations should be understood as linearized):

$$\Phi_{a,a} = 0\ (D \neq 4),\ 3\Lambda_{abd,d} = \Phi_{a,b} - 2\Phi_{b,a},\ \Lambda_{a[bc,d],d} \equiv 0 \ \ \Rightarrow\ \ 3\Lambda_{abc,dd} = -2 f_{bc,a}\,.$$

The last D'Alembert equation has the *source* in its RHS. Some components of Λ (most symmetrical irreducible parts) do not grow (as well as curvature), because (linearized equations)

$$S_{abc,dd} = 0,\ \Phi_{a,dd} = 0,\ \ f_{ab,dd} = 0,\ R_{abcd,ee} = 0.$$

However the least symmetrical Λ-components do go up with time (three growing but powerless polarizations), if the ponderable waves (three f-polarizations) do not vanish; this should be the case for solutions of general position. Again, nothing is unreal!

2.3 Expanding O_4-symmetrical solutions and cosmology

The unique symmetry of AP equations gives scope for symmetrical solutions. In contrast to GR, Eqn. (3) has non-stationary spherically symmetric solutions. The O_4-symmetric field can be generally written [4] as

$$h^a{}_\mu(t, x^i) = \begin{pmatrix} a & bn_i \\ cn_i & en_in_j + d\Delta_{ij} \end{pmatrix}; \quad i,j = (1,2,3,4), \ n_i = \frac{x^i}{r}. \tag{10}$$

Here $a, \ldots, e$ are functions of time, $t = x^0$, and radius r, $\Delta_{ij} = \delta_{ij} - n_in_j$, $r^2 = x^ix^i$. As functions of radius, b, c are odd, while the others are even; other boundary conditions: $e = d$ at $r = 0$, and $h^a{}_\mu \to \delta^a_\mu$ as $r \to \infty$. Placing in (10) $b = 0, e = d$ (the other interesting choice is b=c=0) and making integrations one can arrive to the next system (resembling dynamics of Chaplygin gas; dot and prime denote derivation on time and radius, resp.)

$$A^{\cdot} = AB' - BA' + \frac{3}{r}AB, \ B^{\cdot} = AA' - BB' - \frac{2}{r}B^2, \text{where } A = \frac{a}{e} = e^{1/2}, \ B = -\frac{c}{e}. \tag{11}$$

This system has non-stationary solutions, and a single-wave solution (of proper 'sign') might serve as a suitable (stable) cosmological expanding background. The condition $f_{\mu\nu}$=0 is a must for solutions with such a high symmetry (as well as $S_{\mu\nu\lambda}$=0); so, these O_4-solutions carry no energy, weight nothing—some lack of *gravity* !

More realistic cosmological model might look like a single O_4-wave (or a sequence of waves) moving along the radius and being filled with chaos, or stochastic waves, both powerful (*weak*, $\Delta h \ll 1$) and powerless ($\Delta h < 1$, but intense enough that to give non-linear fluctuations with $\Delta h \sim 1$). The development and examination of stability of this model is an interesting problem. The inhomogeneity of metric in giant O_4-wave can serve as a time-dependent 'shallow dielectric guide' for that weak $f-$waves. The ponderable waves (which slow down the large wave) should have wave-vectors almost tangent to the S^3-sphere of wave-front to be trapped inside this shallow wave-guide; the imponderable waves can

grow up, and partly escape from the wave-guide, and their wave-vectors can be less tangent to the S^3-sphere. The waveguide thickness can be small for an 'observer' in the center of O_4-symmetry, but in co-moving coordinates it can be very large (still $\ll R$). This model can explain well the SNe1a redshift data [4].

References

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